Problem Set 9 due November 18, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: For any angle α, consider the complex number:

$$z = \cos\alpha + i\sin\alpha$$

(1) Compute the product of z with the complex number $z' = \sin\alpha + i\cos\alpha$. Simplify as much as possible! Draw z, z' and zz' on a picture of the complex plane. *(10 points)*

(2) Compute the product of z with the complex number $w = \cos\beta + i\sin\beta$ for any angle β, using the polar form of z and w. Simplify as much as possible! *(5 points)*

(3) Use your result from part (2) to obtain formulas for:

$$\cos(\alpha+\beta) = \ldots$$
$$\sin(\alpha+\beta) = \ldots$$

(5 points)

Problem 2: The damped harmonic oscillator (a.k.a. mass on a spring, moving in a straight line in the presence of friction) obeys the following linear differential equation:

$$\underbrace{1}_{\text{mass}} x''(t) + \underbrace{2}_{\text{friction coefficient}} x'(t) + \underbrace{2}_{\text{spring constant}} x(t) = 0$$

(this is math, so no need to assign units to the numbers above). The initial position is $x(0) = 0$ and the initial velocity is $x'(0) = 1$. Find the complete solution $x(t)$ to the second order differential equation above by converting it into a system of two first order differential equations. Write your answer both in terms of complex exponentials <u>and</u> sines and cosines, by converting from one to the other using formula (223) of the lecture notes. *(20 points)*

Problem 3: Write the symmetric matrix:

$$S = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}$$

explicitly as $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal. Explain all of your steps (*Hint: the characteristic polynomial of a* 4×4 *matrix is a degree 4 polynomial, and therefore difficult in general to solve; however, in the case at hand, it will be easily possible to find its roots*) *(20 points)*

Problem 4: Let S be a symmetric matrix. Use the fact that $S = Q\Lambda Q^T$, where Q is orthogonal and Λ is the diagonal matrix of eigenvalues, to prove that any diagonal entry of S lies between the smallest and the largest eigenvalue of S. (*Hint: write out the diagonal entries of* S *explicitly in terms of the entries of* Q) *(15 points)*

Problem 5: Consider the 3×3 symmetric matrix S such that:

$$\begin{bmatrix} x & y & z \end{bmatrix} S \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - y + 2z)^2$$

for any x, y, z.

(1) Without doing any computations on S, explain why S cannot have full rank. *(5 points)*

(2) Write S out explicitly. *(5 points)*

(3) Compute the eigenvalues and eigenvectors of S. *(10 points)*

(4) Does your answer in part (3) agree with part (1)? Is S positive definite, positive semi-definite, or neither? *(5 points)*